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ON A FUNDAMENTAL THEOREM IN TRIGONOMETRY.

By G. A. MILLER.

Many of our elementary text-books on trigonometry give the values of the trigonometric functions of the following eight angles: $\pm x$, $90 \pm x$, $180 \pm x$, $270 \pm x$. It has been observed that these eight angles may be derived from any one of them by means of the operations of a well known group which is known as the octic group, or the group of the square.[*] The present note is intended to exhibit more clearly the value of this point of view by presenting the matter in a somewhat different form from the one given in the article to which reference has been made.

We shall represent by c the operations of taking the complement, while p will be used to represent that the angle is increased by 90°. As angles which differ as a multiple of 360° are considered identical these operations are of periods 2 and 4, respectively. In fact the first is of period 2 independently of the modulus. The operations of increasing an angle by 180° and by 270° will be denoted bp p^2 and p^3, respectively, since these results are obtained by increasing the angle twice or three times by 90°. In accord with this $p^5 = p$, $c^3 = c$, $p^4 = c^2 = 1$.

The eight operations of the octic group may be written as follows:

$$\begin{array}{cccc} 1 & p & p^2 & p^3 \\ c & cp & cp^2 & cp^3 \end{array}$$

If these operations are performed on the angle x the resulting angles are respectively:

$$\begin{array}{cccc} x & x+90^\circ & x+180^\circ & x+270^\circ \\ 90^\circ - x & 180^\circ - x & 270^\circ - x & -x \end{array}$$

By means of these results the functions of the eight angles mentioned above may be expressed in terms of functions of x provided we know the functions of $cx = 90 - x$, and of $px = x + 90^\circ$. The former functions are simply a matter of definition while the latter are given in the following table:

$$\sin px = \sin(90^\circ + x) = \cos x$$
$$\cos px = \cos(90^\circ + x) = -\sin x$$
$$\tan px = \tan(90^\circ + x) = -\cot x$$
$$\cot px = \cot(90^\circ + x) = -\tan x$$
$$\sec px = \sec(90^\circ + x) = -\csc x$$
$$\csc px = \csc(90^\circ + x) = \sec x.$$

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[*] *Quarterly Journal of Mathematics*, Vol. 37 (1906), p. 226.

Hence,

$\sin p^2x=\sin(180^\circ+x)=\cos px=-\sin x$[*]
$\cos p^2x=\cos(180^\circ+x)=-\sin px=-\cos x$
$\tan p^2x=\tan(180^\circ+x)=-\cot px=\tan x$
$\cot p^2x=\cot(180^\circ+x)=-\tan px=\cot x$
$\sec p^2x=\sec(180^\circ+x)=-\csc px=-\sec x$
$\csc p^2x=\csc(180^\circ+x)=\sec px=-\csc x.$

$\sin p^3x=\sin(270^\circ+x)=\cos p^2x=-\cos x$[†]
$\cos p^3x=\cos(270^\circ+x)=-\sin p^2x=\sin x$
$\tan p^3x=\tan(270^\circ+x)=-\cot p^2x=-\cot x$
$\cot p^3x=\cot(270^\circ+x)=-\tan p^2x=-\tan x$
$\sec p^3x=\sec(270^\circ+x)=-\csc p^2x=\csc x$
$\csc p^3x=\csc(270^\circ+x)=\sec p^2x=-\sec x.$

$\sin cpx=\sin(180^\circ-x)=\cos cx=\sin x$
$\cos cpx=\cos(180^\circ-x)=-\sin cx=-\cos x$
$\tan cpx=\tan(180^\circ-x)=-\cot cx=-\tan x$
$\cot cpx=\cot(180^\circ-x)=-\tan cx=-\cot x$
$\sec cpx=\sec(180^\circ-x)=-\csc cx=-\sec x$
$\csc cpx=\csc(180^\circ-x)=\sec cx=\csc x.$

$\sin cp^2x=\sin(270^\circ-x)=-\sin cx=-\cos x$[‡]
$\cos cp^2x=\cos(270^\circ-x)=-\cos cx=-\sin x$
$\tan cp^2x=\tan(270^\circ-x)=\tan cx=\cot x$
$\cot cp^2x=\cot(270^\circ-x)=\cot cx=\tan x$
$\sec cp^2x=\sec(270^\circ-x)=-\sec cx=-\csc x$
$\csc cp^2x=\csc(270^\circ-x)=-\csc cx=-\sec x.$

$\sin cp^3x=\sin -x=-\cos cx=-\sin x$
$\cos cp^3x=\cos -x=\sin cx=\cos x$
$\tan cp^3x=\tan -x=-\cot cx=-\tan x$
$\cot cp^3x=\cot -x=-\tan cx=-\cot x$
$\sec cp^3x=\sec -x=\csc cx=\sec x$
$\csc cp^3x=\csc -x=-\sec cx=-\csc x$

The advantages of the present view point result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group we know directly what two angles may be made fundamental in the study of the others. The necessary and sufficient condition is that the two angles correspond to generators in the octic group. In other words they must correspond to two non-commutative operators of this group since any two non-commutative operators of a non-abelian group of order p^3 generate the group and any two generators are non-commutative.

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[*] Sin $p^2x=\sin pz=\cos z$, where $z=px$.

[†] Sin $p^3x=\sin py=\cos y$, where $y=p^2x$; also, $\sin p^3x=\sin p^2w=-\sin w$, where $w=px$

[‡] Also, $\sin cp^2x=\sin py=\cos y$, where $y=cpx$, etc.

The properties of the octic group suggest an almost endless number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them can be expressed in terms of the functions of any other one of them just as easily as in terms of functions of x. Since the octic group has only one invariant operator besides the identity the corresponding angle cannot be used as one of two fundamental angles, while all the other angles besides the one corresponding to the identity can be used as one of these two angles. If x is the angle corresponding to the identity $180^\circ+x$ corresponds to the invariant operator of order 2. If this angle corresponds to the identity while c and p have the same meaning as before the eight operators of the octic group in the third paragraph correspond respectively to the following angles:

$$\begin{array}{cccc} 180^\circ+x & 270^\circ+x & x & 90^\circ+x \\ 270^\circ-x & -x & 90^\circ-x & 180^\circ-x \end{array}$$

Hence, for example,

$$\begin{array}{l}
\sin p^2(180^\circ+x)=\sin x=\cos p(180^\circ+x)=-\sin(180^\circ+x) \\
\cos p^2(180^\circ+x)=\cos x=-\sin p(180^\circ+x)=-\cos(180^\circ+x) \\
\tan p^2(180^\circ+x)=\tan x=-\cot p(180^\circ+x)=\tan(180^\circ+x) \\
\cot p^2(180^\circ+x)=\cot x=-\tan p(180^\circ+x)=\cot(180^\circ+x) \\
\sec p^2(180^\circ+x)=\sec x=-\csc p(180^\circ+x)=-\sec(180^\circ+x) \\
\csc p^2(180^\circ+x)=\csc x=\sec p(180^\circ+x)=-\csc(180^\circ+x).
\end{array}$$

The use of this octic group makes the work entirely analytic after the functions of the fundamental angles are known. The main interest however lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this view point even if he did not consider it wise to present it to the beginner.